

Inequalities for Zero-Balanced Gaussian hypergeometric function[★]

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Abstract: In this paper, we consider the monotonicity of certain combinations of the Gaussian hypergeometric functions $F(a-1, b; a+b; 1-x^c)$ and $F(a-1-\delta, b+\delta; a+b; 1-x^d)$ on $(0, 1)$ for $\delta \in (a-1, 0)$, and study the problem of comparing these two functions, thus get the largest value $\delta_1 = \delta_1(a, c, d)$ such that the inequality $F(a-1, b; a+b; 1-x^c) < F(a-1-\delta, b+\delta; a+b; 1-x^d)$ holds for all $x \in (0, 1)$.

Key Words: Gaussian hypergeometric function, monotonicity, inequality.

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1 Introduction

In this paper we consider the Gaussian hypergeometric function

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad (1.1)$$

for $x \in (-1, 1)$, where (a, n) denotes the shifted factorial function $(a, n) \equiv a(a+1) \cdots (a+n-1)$, $n = 1, 2, \dots$, and $(a, 0) = 1$ for $a \neq 0$. It is well known that the function $F(a, b; c; x)$ has many important applications in geometric function theory, theory of mean values, and in several other contexts, and many classes of elementary functions and special functions in mathematical physics are particular or limiting cases of this function [2–7, 9, 11–13].

For $r \in (0, 1)$ and $a \in (0, 1)$, the generalized elliptic integrals of the first and second kinds are defined as

$$\mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2), \quad \mathcal{E}_a(r) = \frac{\pi}{2} F(a-1, 1-a; 1; r^2).$$

In the particular case $a = 1/2$, the generalized elliptic integrals reduce to the complete elliptic integrals

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right).$$

J. M. Borwein and P. B. Borwein, in order to find out the connections between the arithmetic-geometric mean value and other mean values, showed in their paper [7] that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right) < F\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; 1-x^3\right),$$

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for $\delta = 1/6$ and $x \in (0, 1)$.

Subsequently, it was proved by Anderson et al. in [2] that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c\right) < F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d\right) < F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^d\right), \quad (1.2)$$

for all $x \in (0, 1)$, $c, d \in (0, \infty)$ with $0 < 4c < \pi d < \infty$ and $\delta \in (0, \delta_0)$ where $\delta_0 = [(d\pi - 4c)/(4\pi d)]^{\frac{1}{2}}$. It was conjectured for $c = 2, d = 3$ that the best value of δ_0 for which (1.2) is valid is

$$\delta_0 = \frac{1}{\pi} \arccos \frac{2}{3} \approx 0.268.$$

In [4], Anderson et al. considered the more general case of (1.2). They showed several monotonicity theorems of certain combinations of $F(a, b; a+b; 1-x^c)$ and $F(a-\delta, b+\delta; a+b; 1-x^d)$ on $(0, 1)$ for given $a, b, c, d \in (0, \infty)$, $a \leq b$ and $c \leq d$, and found $\sup\{\delta \in (0, a) | F(a, b; c; 1-x^c) < F(a-\delta, b+\delta; a+b; 1-x^d) \text{ for } x \in (0, 1)\}$. Thus the above conjecture and the following open problem raised in [4] were answered.

Open problem. Is it true, for small values of δ , say $0 < \delta < \min\{a, b\}$, that

$$F(a, b; a+b; 1-x^c) < F(a-\delta, b+\delta; a+b; 1-x^d),$$

for $x \in (0, 1)$, $a, b, c, d \in (0, \infty)$ with $0 < c < d < \infty$?

Motivated by the results mentioned above, the following question was naturally raised.

Question. What is the best value of $\delta_1 = \delta(a, c, d) \in (a-1, 0)$ such that

$$F(a-1, b; a+b; 1-x^c) < F(a-1-\delta, b+\delta; a+b; 1-x^d),$$

for $x \in (0, 1)$, $a \in (0, 1)$, $b \geq 1-a$ and $0 < c < d < \infty$.

In [15], Song et al. established a monotonicity theorem of certain combinations of $F(-1/2, 1/2; 1; 1-x^c)$ and $F(-1/2-\delta, 1/2+\delta; 1; 1-x^d)$ on $(0, 1)$ for given $0 < c \leq 5d/6$, and got the following inequality: For $\delta_1 = (\sqrt{c/d} - 1)/2$ and $\delta \in (-1/2, \delta_1)$,

$$F\left(-\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c\right) < F\left(-\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d\right). \quad (1.3)$$

and $\delta_1 = (\sqrt{c/d} - 1)/2$ is the largest value for the inequality (1.3) holds for all $x \in (0, 1)$,

Besides, they also considered monotonicity property of certain combinations of $F(a-1-\delta, 1-a+\delta; 1; 1-x^3)$ and $F(a-1, 1-a; 1; 1-x^2)$ for given $a \in [1/29, 1)$ and $\delta \in (a-1, 0)$, and found the largest value δ_1 such that inequality $F(a-1, 1-a; 1; 1-x^2) < F(a-1-\delta, 1-a+\delta; 1; 1-x^3)$ holds for all $x \in (0, 1)$.

In this paper, we will show a monotonicity theorem of certain combinations of $F(a-1, b; a+b; 1-x^c)$ and $F(a-1-\delta, b+\delta; a+b; 1-x^d)$ on $(0, 1)$, and find the largest value $\delta_1 = \delta_1(a, c, d)$ such that the inequality $F(a-1, b; a+b; 1-x^c) < F(a-1-\delta, b+\delta; a+b; 1-x^d)$ holds for all $x \in (0, 1)$. Throughout this paper, we shall always let $a \in (0, 1)$, $b \geq 1-a$, and

$$\begin{aligned} \alpha &= a(b+1), \quad \beta = b(1-a), \quad p = \alpha + \beta = a+b, \\ h &= \alpha\beta(p+\beta) = a(1-a)b(b+1)(a+2b-ab), \\ k &= \beta(p+1) + p = b(1-a)(a+b+1) + a+b. \end{aligned}$$

The main results are stated as follows.

Theorem 1.1. Assume that $a \in (0, 1)$, $b \geq 1-a$, and α, β, p, h satisfy either $\alpha \geq \sqrt{3}\beta$ or $\alpha < \sqrt{3}\beta$, and $4h(\beta+p) \geq p^4$. Let $0 < c/d \leq (\beta+p)/k$, and δ_1 be the large root of $(c/d-1)\beta + (a-b-1)\delta - \delta^2 = 0$, namely $\delta_1 = [(a-b-1) + \sqrt{(p-1)^2 + 4\beta c/d}]/2 < 0$. We have that

(1) If $\delta \in (a-1, \delta_1]$, the function

$$G(x) = \frac{F(a-1-\delta, b+\delta; p; 1-x^d) - F(a-1, b; p; 1-x^c)}{1-x^c}$$

is strictly decreasing from $(0, 1)$ onto $(C_1(\delta), C_2(\delta))$, where

$$C_1(\delta) = \frac{d}{pc} \left(\left(\frac{c}{d} - 1 \right) \beta + (a - b - 1) \delta - \delta^2 \right) \geq 0$$

$$C_2(\delta) = \frac{1}{pB(a - \delta, b + 1 + \delta)} - \frac{1}{pB(a, b + 1)}.$$

In particular, for all $x \in (0, 1)$ and $\delta \in (a - 1, \delta_1]$,

$$F(a - 1, b; p; 1 - x^c) + C_1(\delta)(1 - x^c) < F(a - 1 - \delta, b + \delta; p; 1 - x^d) \\ < F(a - 1, b; p; 1 - x^c) + C_2(\delta)(1 - x^c).$$

(2) If $\delta_1 < \delta < 0$, as the functions of x , $F(a - 1 - \delta, b + \delta; p; 1 - x^d)$ and $F(a - 1, b; p; 1 - x^c)$ are not directly comparable on $(0, 1)$, that is, neither

$$F(a - 1, b; p; 1 - x^c) < F(a - 1 - \delta, b + \delta; p; 1 - x^d),$$

nor its reversed inequality holds for all $x \in (0, 1)$.

The following Theorem can be directly obtained by Theorem 1.1.

Theorem 1.2. Assume that $a \in (0, 1)$, $b \geq 1 - a$. Let α, β, p, h be as in Theorem 1.1, if $0 < c/d \leq (\beta + p)/k$, then

$$\sup\{\delta \in (a - 1, 0) | F(a - 1, b; a + b; 1 - x^c) < F(a - 1 - \delta, b + \delta; a + b; 1 - x^d), \\ \text{for all } x \in (0, 1)\} = \frac{a - b - 1 + \sqrt{(p - 1)^2 + 4\alpha c/d}}{2}.$$

2 Preliminaries

Before we prove our main results stated in Section 1, we need to establish several technical lemmas. Firstly, let us recall some known results for $F(a, b; c; x)$ and for the gamma function.

For $x > 0, y > 0$, the Euler gamma function $\Gamma(x)$, its logarithmic derivative $\Psi(x)$ and the beta function $B(x, y)$ are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

respectively (c.f.[16]). The gamma function satisfies the difference equation ([16], p. 237)

$$\Gamma(x + 1) = x\Gamma(x),$$

if x is not a nonpositive integer and has the so-called reflection property ([16], p. 239)

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}. \quad (2.1)$$

We shall also need an asymptotic formula of gamma function ([14], p. 628)

$$\frac{\Gamma(n + a)}{\Gamma(n + b)} \sim n^{a-b}, \quad n \rightarrow +\infty, n \in \mathbb{N}. \quad (2.2)$$

The hypergeometric function (1.1) has the following difference formula ([14]),

$$\frac{dF(a, b; c; x)}{dx} = \frac{ab}{c} F(a + 1, b + 1; c + 1; x)$$

and the asymptotic limit ([14], p. 630),

$$\lim_{x \rightarrow 1^-} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad c > a + b.$$

The following Lemma can be find Lemma 2.1.5 in [13], and Lemma 2.11 in [4], respectively.

Lemma 2.1. (1) For $a, b, c, d \in (0, \infty)$, the function $x \rightarrow (1-x)^d F(a, b; c; x)$ is (strictly) decreasing on $(0, 1)$ if and only if $d \geq \max\{a+b-c, ab/c\}$ ($d > \max\{a+b-c, ab/c\}$).

(2) For $a, b \in (0, \infty)$ with $a \leq b$, the function $x \rightarrow B(a-x, b+x)$ is strictly increasing and convex on $(0, a)$.

Lemma 2.2. Let $a \in (0, 1)$, $b \geq 1-a$, and α, β, p, h, k be as in Section 1, a function $g_1(y)$ is defined as

$$g_1(y) = y^2 + \frac{p^2}{k}y + \frac{h(p+\beta)}{k^2}.$$

Then, (1) if α and β satisfy $\alpha \geq \sqrt{3}\beta$, $g_1(y)$ is an increasing function from $(-h/(\alpha k), 0)$ onto $((1-a)^2(p^2 + p\beta)/k^2, h/k^2)$.

(2) If α, β, p and h satisfy $\alpha < \sqrt{3}\beta$, and $4h(\beta+p) \geq p^4$, we have $g_1(y) \geq 0$ for all $(-h/(\alpha k), 0)$.

Proof. (1) Clearly,

$$g_1(-h/(\alpha k)) = (1-a)^2(p^2 + p\beta)/k^2 > 0, \quad g_1(0) = h/k^2.$$

Since $\alpha \geq \sqrt{3}\beta$,

$$\frac{p^2}{2k} - \frac{h}{\alpha k} = \frac{\alpha^2 - 3\beta^2}{2k} \geq 0,$$

hence, for $y \in (-h/(\alpha k), 0)$,

$$g'_1(y) = 2y + p^2/k = \frac{p^2}{k} - \frac{2h}{\alpha k} \geq 0,$$

and $g_1(y)$ is an increasing function.

(2) For $\alpha < \sqrt{3}\beta$, and $4h(\beta+p) \geq p^4$, we have

$$g_1(y) = \left(y + \frac{p^2}{2k}\right)^2 + \frac{h(\beta+p)}{k^2} - \frac{p^4}{4k^2} \geq 0.$$

hence, $g_1(y) \geq 0$ for all $(-h/(\alpha k), 0)$. □

Remark 2.3. Let $a \in (0, 1)$, $b \geq 1-a$, and α, β, p and h be as in Section 1, we have that

$$(1) \quad \alpha \leq \sqrt{3}\beta \Leftrightarrow \sqrt{3}/a - 1/b \geq 1 + \sqrt{3}.$$

$$(2) \quad \alpha < \sqrt{3}\beta, \quad 4h(\beta+p) \geq p^4 \Leftrightarrow \sqrt{3}/a - 1/b > 1 + \sqrt{3},$$

$$4a(1-a)b(b+1)(a+2b-ab) \geq (a+b)^4.$$

Lemma 2.4. Let $a \in (0, 1)$, $b \geq 1-a$, α, β, p and h be as in Section 1, and $D = \{(x, y) | 0 < x < (\beta+p)/k, -\beta x < y < 0\}$. Define the function $g(x, y)$ on the domain D as

$$g(x, y) = y^2 + ((p+1)x - 1)y + \alpha\beta x^2.$$

If α, β, p and h satisfy either $\alpha \geq \sqrt{3}\beta$, or $\alpha < \sqrt{3}\beta$ and $4h(\beta+p) \geq p^4$, then $\inf_{(x,y) \in h} g(x, y) = 0$.

Proof. By differentiation,

$$\frac{\partial g(x, y)}{\partial x} = (p+1)y + 2\alpha\beta x, \quad \frac{\partial g(x, y)}{\partial y} = 2y + (p+1)x + 1.$$

Let $\partial g(x, y)/\partial x = \partial g(x, y)/\partial y = 0$, we have

$$x_0 = \frac{p+1}{(p+1)^2 - 4\alpha\beta}, \quad y_0 = -\frac{2\alpha\beta}{(p+1)^2 - 4\alpha\beta}, \quad g(x_0, y_0) = \frac{\alpha\beta}{(p+1)^2 - 4\alpha\beta} > 0.$$

On the other hand,

$$\begin{aligned} g(x, 0) &= \alpha\beta x^2 \geq 0 & \text{for } 0 < x < (\beta + p)/k, \\ g(x, -\beta x) &= \beta x(1 - x) > 0 & \text{for } 0 < x < (\beta + p)/k < 1. \end{aligned}$$

Since

$$g((\beta + p)/k, y) = y^2 + \frac{p^2}{k}y + \frac{h(p + \beta)}{k^2} = g_1(y), \quad y \in (-h/(\alpha k), 0),$$

we get $g((\beta + p)/k, y) \geq 0$ for all $y \in (-h/(\alpha k), 0)$ by Lemma 2.2, hence $\inf_{(x,y) \in h} g(x, y) = 0$. \square

Since $b \geq 1 - a$, we have the following Lemma.

Lemma 2.5. *Let $a \in (0, 1)$, $b \geq 1 - a$, $u = a - \delta$, $v = a + \delta$ and $\delta \in (a - 1, 0)$, we have*

(1) *the function $f_1(\delta) = uv + u - 1 + \beta = p - 1 + (a - b - 1)\delta - \delta^2$ is strictly decreasing from $(a - 1, 0)$ onto $(p - 1, p - 1 + \alpha)$.*

(2) *the function $f_2(\delta) = u(v + 1) = \alpha + (a - b - 1)\delta - \delta^2$ is strictly decreasing from $(a - 1, 0)$ onto (α, p) .*

(3) *the function $f_3(\delta) = v(u - 1) = -\alpha + (a - b - 1)\delta - \delta^2$ is strictly decreasing from $(a - 1, \delta_1)$ onto $(-c\beta/d, 0)$, where δ_1 is as in Theorem 1.1.*

Lemma 2.6. *The function $f_4(a) = 4a(2 - a)(1 - a)^2(a^2 - 2a + 2)^2 - 1$ has only two null points $a_0 \in (1/32, 1/31)$, $a_1 \in (41/50, 42/50)$ in $(0, 1)$.*

Proof. Since $f_4(0) = -1$, $f_4(1/2) = 299/64$, $f_4(1) = -1$ and $f_4(x)$ has at least two null points in $(0, 1)$. Assume that $f_4(x)$ has more than two null points in $(0, 1)$, then $f_4'(x)$ has more than two null points in $(0, 1)$ by Rolle mean value theorem. But,

$$f_4'(a) = -8(a - 1)(a^2 - 2a - 2)(4a^4 - 16a^3 + 15a^2 + 2a - 2),$$

$a - 1 < 0$, $a^2 - 2a - 2 < 0$, it is easy to know that $f_5(a) = 4a^4 - 16a^3 + 15a^2 + 2a - 2$ is an increasing function in $(0, 1)$, $f_5(0) = -2$ and $f_5(1) = 3$, hence $f_5(a)$ has only one root in $(0, 1)$, Contradiction. By elementary computations, $f_4(1/32) < 0$, $f_4(1/31) > 0$, $f_4(41/50) > 0$ and $f_4(42/50) < 0$, so there exist two null points $a_0 \in (1/32, 1/31)$ and $a_1 \in (41/50, 42/50)$ in $(0, 1)$. \square

Lemma 2.7. *If $a \in (0, 1)$, $b \geq 1 - a$, $0 < c/d \leq (\beta + p)/k$, $\delta \in (a - 1, 0)$ and $n \in \mathbb{N}$, let $u = a - \delta$, $v = b + \delta$, then*

$$Q(n) = \frac{\Gamma(u + n - 1)\Gamma(v + n)}{\Gamma(a + n - 1)\Gamma(b + n)} \left\{ \left(\frac{c}{d} - 1 \right) (u + v + n) + u(v + 1) \right\}$$

is strictly decreasing and $\lim_{n \rightarrow \infty} Q(n) = -\infty$.

Proof. By computation, we have

$$Q(n + 1) - Q(n) = \frac{\Gamma(n + u - 1)\Gamma(n + v)}{\Gamma(n + a)\Gamma(b + n + 1)} Q_1(n),$$

where

$$\begin{aligned} Q_1(n) &= (c/d - 1)n^2 - (c/d - 1)(uv + u - 1 + \beta)n + A = (c/d - 1)n^2 - (c/d - 1)f_1(\delta)n + A, \\ A &= (c/d - 1)(u + v + 1) + u(v + 1)v(u - 1) - \beta((c/d - 1)(u + v) + u(v + 1)). \end{aligned}$$

Since $\delta \in (a - 1, 0)$ and $f_1(\delta) \geq f_1(0) = p - 1 \geq 0$. Hence, $Q_1(n)$ is strictly decreasing and

$$\begin{aligned} Q_1(n) &\leq Q_1(1) = (u(v + 1))^2 + [(c/d - 1)(2 + p) - \alpha]u(v + 1) - (c/d - 1)\alpha(p + 1) \\ &= f_2(\delta)^2 + [(c/d - 1)(2 + p) - \alpha]f_2(\delta) - (c/d - 1)\alpha(p + 1) \\ &=: F(f_2(\delta)). \end{aligned}$$

Since $f_2(\delta)$ is strictly decreasing from $(a-1, 0)$ onto (α, p) by Lemma 2.5 and $0 < c/d \leq (\beta+p)/k < 1$, we have

$$F(\alpha) = (c/d - 1)\alpha < 0, \quad F(p) = c/(dk) - (\beta + p) < 0.$$

Hence, it is easy to know that $Q_1(n) < 0$ for $n \in \mathbb{N}$, and the monotonicity of $Q(n)$ follows. Moreover, by (2.2), we have

$$\lim_{n \rightarrow \infty} Q(n) = \lim_{n \rightarrow \infty} \left[\left(\frac{d}{c} - 1 \right) (n + u + v) + u(v + 1) \right] = -\infty.$$

□

Lemma 2.8. For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow R$ be continuous on $[a, b]$, and be differentiable on (a, b) . Let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3 Proof of the main theorem

Proof of Theorem 1.1. Let $u = a - \delta, v = b + \delta$ and $t = 1 - (1 - x)^{d/c}$, we obtain that

$$\begin{aligned} G_1(x) &= G((1 - x)^{1/c}) = \frac{1}{x} [F(a - 1 - \delta, b + \delta; p; t) - F(a - 1, b; p; x)] \\ &= \frac{1}{x} [F(u - 1, v; p; t) - F(a - 1, b; p; x)], \end{aligned} \quad (3.1)$$

we let $f(x) = F(u - 1, v; p; t) - F(a - 1, b; p; x)$ and $g(x) = x$, then $G_1(x) = f(x)/g(x)$ and $f(0) = g(0) = 0$.

$$\begin{aligned} \frac{f'(x)}{g'(x)} &= f'(x) = \frac{d}{c} (1 - x)^{(d/c)-1} \frac{v(u-1)}{p} F(u, v+1; p+1; t) \\ &\quad + \frac{\beta}{p} F(a, b+1; p+1; x), \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} f''(x) &= -\frac{v(u-1)d}{cp} \left(\frac{d}{c} - 1 \right) (1 - x)^{(d/c)-2} F(u, v+1; p+1; t) \\ &\quad + \frac{u(u-1)v(v+1)d^2}{p(p+1)c^2} (1 - x)^{2[(d/c)-1]} F(u+1, v+2; p+1, t) \\ &\quad + \frac{\alpha\beta}{p(p+1)} F(a+1, b+2; p+2; x). \end{aligned} \quad (3.3)$$

The desired monotonicity of $G_1(x)$ will follow from Lemma 2.8 if we can prove that $f'(x)$ is increasing on $(0, 1)$ or $f''(x) > 0$ on $(0, 1)$. It is easy to know that $x \rightarrow (1 - x)^{1/c} (c > 0)$ is strictly decreasing on $(0, 1)$. Let

$$\begin{aligned} h(t) &= -\frac{v(u-1)d}{cp} \left(\frac{d}{c} - 1 \right) (1 - t) F(u, v+1; p+1; t) \\ &\quad + \frac{u(u-1)v(v+1)d^2}{p(p+1)c^2} (1 - t)^2 F(u+1, v+2; p+2, t). \end{aligned}$$

Since $(1 - t) = (1 - x)^{(d/c)}$, then it follows from (3.3) that

$$(1 - x)^2 f''(x) = h(t) + \frac{\alpha\beta}{p(p+1)}(1 - x)^2 F(a+1, b+2; p+2; x). \quad (3.4)$$

Using the series expansion for $F(a, b; c; x)$, we have

$$\begin{aligned} h(t) &= \frac{d^2}{c^2}(1 - t) \left[\left(\frac{c}{d} - 1 \right) \frac{v(u-1)}{p} \sum_{n=0}^{\infty} \frac{(u, n)(v+1, n)}{(p+1, n)} \frac{t^n}{n!} \right. \\ &\quad \left. + (1 - t) \frac{u(u-1)v(v+1)}{p(p+1)} \sum_{n=0}^{\infty} \frac{(u+1, n)(v+2, n)}{(p+2, n)} \frac{t^n}{n!} \right], \\ &= \frac{d^2}{c^2}(1 - t) \left[\left(\frac{c}{d} - 1 \right) \sum_{n=0}^{\infty} \frac{(u-1, n+1)(v, n+1)}{(p, n+1)} \frac{t^n}{n!} \right. \\ &\quad \left. + (1 - t) \sum_{n=0}^{\infty} \frac{(u-1, n+2)(v, n+2)}{(p, n+2)} \frac{t^n}{n!} \right], \\ &= \frac{d^2}{c^2}(1 - t) \sum_{n=0}^{\infty} \frac{(u-1, n+1)(v, n+1)}{(p, n+2)} \left[\left(\frac{c}{d} - 1 \right) (p+n+1) \right. \\ &\quad \left. + (u+n)(v+n+1) - n(p+n+1) \right] \frac{t^n}{n!}. \end{aligned} \quad (3.5)$$

Since $a \in (0, 1)$ and $b \geq 1 - a$, $2 > \max\{a+1+b+2-(a+b+2), [(a+1)(b+2)]/(a+b+2)\}$, we have that $(1 - x)^2 F(a+1, b+2; a+b+2; x)$ is strictly decreasing on $(0, 1)$ by Lemma 2.1(1). While $t/x = [1 - (1 - x)^{d/c}]/x$ is strictly decreasing from $(0, 1)$ onto $(1, d/c)$. Thus, $t > x$ and the following inequality holds

$$(1 - x)^2 F(a+1, b+2; a+b+2; x) > (1 - t)^2 F(a+1, b+2; a+b+1; t). \quad (3.6)$$

By the series expansion of $F(a, b; c; x)$, we obtain that

$$\begin{aligned} \frac{\alpha\beta}{p(p+1)}(1 - t)F(a+1, b+2; p+2; t) &= (1 - t) \sum_{n=0}^{\infty} \frac{(a-1, n+2)(b, n+2)}{(p, n+2)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a-1, n+2)(b, n+2)}{(p, n+2)} \frac{t^n}{n!} - \sum_{n=0}^{\infty} \frac{(a-1, n+2)(b, n+2)}{(p, n+2)} \frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a-1, n+2)(b, n+2)}{(p, n+2)} \frac{t^n}{n!} - \sum_{n=0}^{\infty} \frac{n(a-1, n+1)(b, n+1)}{(p, n+1)} \frac{t^n}{n!} \\ &= -\alpha \sum_{n=0}^{\infty} \frac{(a-1, n+1)(b, n+1)}{(p, n+2)} \frac{t^n}{n!}. \end{aligned} \quad (3.7)$$

Hence, it follows from (3.4), (3.5), (3.6), and (3.7) that

$$\begin{aligned} \frac{(1 - x)^2 f''(x)}{1 - t} &> \frac{d^2}{c^2}(1 - t) \sum_{n=0}^{\infty} \frac{(u-1, n+1)(v, n+1)}{(p, n+2)} \left[\left(\frac{c}{d} - 1 \right) (p+n+1) \right. \\ &\quad \left. + (u+n)(v+n+1) - n(p+n+1) \right] \frac{t^n}{n!} - \alpha \sum_{n=0}^{\infty} \frac{(a-1, n+1)(b, n+1)}{(p, n+2)} \frac{t^n}{n!} \\ &= \frac{d^2}{c^2} \sum_{n=1}^{\infty} \frac{(a-1, n)(b, n)}{(p, n+1)} \frac{t^{n-1}}{(n-1)!} \left\{ -\frac{\alpha c^2}{d^2} \right. \\ &\quad \left. + \frac{(u-1, n)(v, n)}{(a-1, n)(b, n)} \left[\left(\frac{c}{d} - 1 \right) (p+n) + (u+n-1)(v+n) - (n-1)(p+n) \right] \right\} \\ &= \frac{d^2}{c^2} \sum_{n=1}^{\infty} \frac{(a, n-1)(b+1, n-1)}{(p, n+1)(n-1)!} \left\{ \frac{\alpha\beta c^2}{d^2} + \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(u-1)\Gamma(v)} Q(n) \right\} t^{n-1}, \end{aligned}$$

where $Q(n)$ is defined as in Lemma 2.7. Since $u - 1 = a - 1 - \delta \in (-1, 0)$, $\Gamma(u - 1) < 0$, it follows from Lemma 2.7 that

$$\begin{aligned} \frac{(1-x)^2 f''(x)}{1-t} &> \frac{d^2}{c^2} \sum_{n=1}^{\infty} \frac{(a, n-1)(b+1, n-1)}{(p, n+1)(n-1)!} \left\{ \frac{\alpha\beta c^2}{d^2} + \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(u-1)\Gamma(v)} Q(1) \right\} t^{n-1} \\ &= \frac{d^2}{c^2} \sum_{n=1}^{\infty} \frac{(a, n-1)(b+1, n-1)}{(p, n+1)(n-1)!} \left[\frac{\alpha\beta c^2}{d^2} + v(u-1)((c/d-1)(p+1) + u(v+1)) \right] t^{n-1} \\ &= g(x, y) \frac{d^2}{c^2} \sum_{n=1}^{\infty} \frac{(a, n-1)(b+1, n-1)}{(p, n+1)(n-1)!} t^{n-1}, \end{aligned} \quad (3.8)$$

where $x = c/d \in (0, (\beta + p)/k]$, $y = f_3(\delta) = (b + \delta)(a - \delta - 1)$, and

$$g(x, y) = y^2 + ((p+1)x - 1)y + \alpha\beta x^2,$$

since $\delta \in (a-1, \delta_1]$, $y \in (-\beta x, 0]$ by Lemma 2.5, it follows from Lemma 2.4 that $g(x, y) \geq 0$ for $(x, y) \in D$, where D is as Lemma 2.4.

Hence, it follows from (3.8) that $f''(x) > 0$ for all $x \in (0, 1)$, which shows that $f'(x)$ is strictly increasing on $(0, 1)$, and so is $G_1(x)$ by (3.1), (3.2) and Lemma 2.8. Moreover, by L'Hôpital's rule, we get

$$G(1^-) = G_1(0^+) = f'(0) = \frac{d}{pc} \left((c/d-1)\alpha + (a-b-1)\delta - \delta^2 \right) = C_1(\delta) \quad (3.9)$$

for $\delta \in (a-1, \delta_1]$, $C_1(\delta) \geq C_1(\delta_1) = 0$ and

$$\begin{aligned} G(0^+) &= G_1(1^-) = f(1^-) = F(a-1-\delta, b+\delta; p; 1) - F(a-1, b; p; 1) \\ &= \frac{1}{pB(a-\delta, b+1+\delta)} - \frac{1}{pB(a, b+1)} = C_2(\delta). \end{aligned} \quad (3.10)$$

For part (2), we observe that, for $\delta_1 < \delta < 0$, the equations (3.9) and (3.10) hold again, both $C_1(\delta)$ and $C_2(\delta)$ are strictly decreasing from Lemma 2.1(3), and $G(1^-) = C_1(\delta) < C_1(\delta_0) = 0$, $G(0^+) = C_2(\delta) > C_2(0^-) = 0$.

Corollary 3.1. *Let a_0 be the minimum root of $4a(2-a)(1-a)^2(a^2-2a+2)^2 = 1$ in $(0, 1)$. For $a \in [a_0, 1)$, $0 < c/d \leq [(a-1)^2 + 1]/[2(a-1)^2 + 1]$, and $\delta_2 = (\sqrt{c/d} - 1)(1-a) < 0$, we have*

(1) $a_0 \in (1/32, 1/31)$,

(2) If $\delta \in (a-1, \delta_2]$, then the function

$$G_1(x) = \frac{F(a-1-\delta, 1-a+\delta; 1; 1-x^d) - F(a-1, 1-a; 1; 1-x^c)}{1-x^c}$$

is strictly decreasing from $(0, 1)$ onto $(C_3(\delta), C_4(\delta))$, where

$$\begin{aligned} C_3(\delta) &= -\frac{d}{c} \left(\delta^2 + 2(1-a)\delta + (1-a)^2 \left(1 - \frac{c}{d} \right) \right) \geq 0 \\ C_4(\delta) &= \frac{1}{B(a-\delta, 2-a+\delta)} - \frac{1}{B(a, 2-a)} = \frac{1}{\pi} \left\{ \frac{\sin(\pi(a-\delta))}{1-a+\delta} - \frac{\sin(\pi a)}{1-a} \right\}. \end{aligned}$$

In particular, for all $x \in (0, 1)$, if $\delta \in (a-1, \delta_2]$,

$$\begin{aligned} F(a-1, 1-a; 1; 1-x^c) + C_3(\delta)(1-x^c) &< F(a-1-\delta, 1-a+\delta; 1; 1-x^d) \\ &< F(a-1, 1-a; 1; 1-x^c) + C_4(\delta)(1-x^c). \end{aligned} \quad (3.11)$$

(3) If $\delta_2 < \delta < 0$, then, as the functions of x , $F(a-1-\delta, 1-a+\delta; 1; 1-x^d)$ and $F(a-1, 1-a; 1; 1-x^c)$ are not directly comparable on $(0, 1)$, that is, neither

$$F(a-1, 1-a; 1; 1-x^c) < F(a-1-\delta, 1-a+\delta; 1; 1-x^d)$$

nor its reversed inequality holds for all $x \in (0, 1)$.

Proof. (1) Part (1) follows from Lemma 2.6.

(2) Let $b = 1 - a$, for $\alpha \geq \sqrt{3}\beta \Leftrightarrow$

$$\frac{\sqrt{3}}{a} - \frac{1}{1-a} \leq 1 + \sqrt{3} \Leftrightarrow a \in (1 - \frac{1}{\sqrt{1+\sqrt{3}}}, 1],$$

for $\alpha < \sqrt{3}\beta$, $4h(\beta + p) \geq p^4 \Leftrightarrow$

$$\begin{aligned} \frac{\sqrt{3}}{a} - \frac{1}{1-a} > 1 + \sqrt{3}, 4a(2-a)(1-a)^2(a^2 - 2a + 2)^2 \geq 1 \\ \Leftrightarrow a \in (0, 1 - \frac{1}{\sqrt{1+\sqrt{3}}}) \cap (a_0, a_1) \Leftrightarrow (a_0, 1 - \frac{1}{\sqrt{1+\sqrt{3}}}), \end{aligned}$$

where a_0, a_1 are as Lemma 2.6. By Theorem 1.1, if $a \in (a_0, 1)$, $0 < c/d \leq [(a-1)^2 + 1]/[2(a-1)^2 + 1]$, and $\delta_1 = (\sqrt{c/d} - 1)(1-a) < 0$, the inequality (3.11) holds.

Part (3) follows from Theorem 1.1(2). \square

Remark 3.2. The following results, which have been proved in [15], can be directly obtained by Corollary 3.1.

(I) Let $a = b = 1/2$, $\alpha = a(b+1) = 3/4$, $\beta = b(1-a) = 1/4$, hence $\alpha > \sqrt{3}\beta$. We have

(1) Let $0 < c/d \leq 5/6$, and $\delta_3 = (\sqrt{c/d} - 1)/2 < 0$. Then, if $\delta \in (-1/2, \delta_3]$, the following inequality holds for all $x \in (0, 1)$,

$$\begin{aligned} F(1/2, 1/2; 1; 1-x^c) + C_3(\delta)(1-x^c) &< F(-1/2 - \delta, 1/2 + \delta; 1; 1-x^d) \\ &< F(-1/2, 1/2; 1; 1-x^c) + C_4(\delta)(1-x^c). \end{aligned}$$

where

$$\begin{aligned} C_5(\delta) &= -\frac{d}{c} \left(\delta^2 + \delta + \frac{1}{4} \left(1 - \frac{c}{d} \right) \right) \geq 0 \\ C_6(\delta) &= \frac{1}{B(1/2 - \delta, 3/2 + \delta)} - \frac{2}{\pi} = \frac{2}{\pi} \left[\frac{\cos(\pi\delta)}{1 + 2\delta} - 1 \right]. \end{aligned}$$

(2) If $0 < c/d \leq 5/6$, then

$$\begin{aligned} \sup\{\delta \in (-1/2, 0) \mid F(1/2, 1/2; 1; 1-x^c) < F(-1/2 - \delta, 1/2 + \delta; 1; 1-x^d), \\ \text{for all } x \in (0, 1)\} = (\sqrt{c/d} - 1)/2. \end{aligned}$$

(II) Let $c = 2, d = 3$ and a_0 be the minimum root of $4a(2-a)(1-a)^2(a^2 - 2a + 2)^2 = 1$. For $a \in (a_0, 1]$, and $\delta_4 = (\sqrt{6}/3 - 1)(1-a) < 0$, we have that:

(1) If $\delta \in (a-1, \delta_4]$, the following inequality hold for all $x \in (0, 1)$,

$$\begin{aligned} F(a-1, 1-a; 1; 1-x^2) + C_7(\delta)(1-x^c) &< F(a-1-\delta, 1-a+\delta; 1; 1-x^3) \\ &< F(a-1, 1-a; 1; 1-x^2) + C_8(\delta)(1-x^c). \end{aligned}$$

where

$$\begin{aligned} C_7(\delta) &= -\frac{3}{2} \left(\delta^2 + 2(1-a)\delta + \frac{(1-a)^2}{3} \right) \geq 0 \\ C_8(\delta) &= \frac{1}{B(a-\delta, 2-a+\delta)} - \frac{1}{B(a, 2-a)} = \frac{1}{\pi} \left[\frac{\sin(\pi(a-\delta))}{1-a+\delta} - \frac{\sin(\pi a)}{1-a} \right]. \end{aligned}$$

(2) If $a \in [a_0, 1)$, then

$$\begin{aligned} \sup\{\delta \in (-1/2, 0) \mid F(a-1, 1-a; 1; 1-x^2) < F(a-1-\delta, 1-a+\delta; 1; 1-x^3), \\ \text{for all } x \in (0, 1)\} = (\sqrt{6}/3 - 1)(1-a). \end{aligned}$$

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